ABOUT TWO OBJECTIONS TO COOK'S PROPOSAL

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Abstract

The main thesis of this work is as follows: there are versions of Yablo's paradox that, if Cook is right about the non-circular character of his version of it, are truly paradoxical and genuinely non-circular, and Cook's version of Yablo's paradox is one of them. Here I will not evaluate the "circular" or "non-circular" side to Cook's proposal. In fact, I think that he is right about it, and that his version of Yablo's list is non-circular. But is it paradoxical? In order to be so, the principles that lead to (i) the derivation of a contradiction, or (ii) the impossibility to give a stable assignment of truth values to the relevant set of sentences, must be acceptable. I will explore two ways to argue that they are not. I will conclude that these attempts lead to a very narrow conception of a theory of truth, or to deny that a paradigmatic case of paradox, such as the "Old-Fashioned Liar," is truly paradoxical.

KEY WORDS: Yablo; Paradox; Truth; Liar.

Resumen

La tesis principal de este trabajo es la siguiente: hay versiones de la paradoja de Yablo tales que, si Cook está en lo cierto acerca del carácter no-circular de su propia versión de ella, son genuinamente paradójicas y auténticamente no-circulares, y la versión de Cook en cuestión es una de ellas. Aquí no voy a evaluar su carácter circular o no-circular. Creo, de hecho, que Cook está en lo correcto sobre el punto. Pero, ¿es su versión auténticamente paradójica? Para que lo fuera, los principios que llevan a (i) derivar una contradicción, o (ii) la imposibilidad de dar una asignación de valores de verdad estables al conjunto relevante de oraciones, deben ser aceptables. Voy a explorar dos modos de argumentar que no lo son. Voy a concluir que estos intentos llevan a una concepción de la teoría de la verdad muy estrecha, o a negar que un caso paradigmático de paradoja, como el "Mentiroso Tradicional", sea auténticamente paradójica.

PALABRAS CLAVE: Yablo; Paradoja; Verdad; Mentiroso.

1. Some arguments for infinitary languages

A way to deny that the paradox formulated in Lp –or its deductive systems D and δD– or in Bringsjord and Heuveln's system is a genuine paradox is to reject infinitary systems, or finitary systems, that make the ω rule –or a version of it, such as the Conjunction Introduction Rule of D– a valid one. But this is not a good idea. A discipline interested in truth preservation should study what happens with them. Truth preservation seems a sufficiently interesting phenomenon, and an inference that preserves truth seems good enough.[1] If Logic is focused on truth preservation, because it is interested in validity, then it should study all systems that preserve truth. Truth preservation (from premises to conclusion) happens not only in classical finitary and bivalent systems, but also in all sorts of finitary and non-bivalent systems. But it also happens in infinitary systems. In Cook's D and δD systems, some of those inferences have the form of the conjunction introduction –the analogue, in Cook's systems, of the ω rule. And they have infinite premises. Cook presents proofs of the soundness of both systems. So, why should we reject these systems?

Here is a possible answer: in Cook's systems, proofs may be infinitely long. This distorts the idea of "proof" as a finite sequence of steps. But, of course, infinitary systems introduce another idea of "proof." A proof, now, is an operation that can be carried out by a machine. The two notions of proofs may be co-extensive, but they are not. A hypercomputer is a machine that could carry out operations involving infinite steps, and henceforth, calculate infinitary proofs.[2] These infinitary "proofs" are not "proofs" in the classical, finitary sense.[3]

What counts as a "proof" might be merely a verbal debate. But there are some things to say in favor of the infinitary sense of "proof": (1) this idea has an obvious resemblance to the classical approach –a "proof" is an operation carried out in a(n) (eventually infinite) number of steps–,

[1] For example, because truth-preserving inferences are a way to acquire knowledge. If I know that an inference preserves truth, and I know that each premise is true, then I know (or at least I am in a position to know) that the conclusion is true. An inference that preserves truth guarantees that, if premises are true, the conclusion is also true. Thus, knowing that gives me an epistemic reassurance.

[2] Arguments for infinitary systems other than the ones presented here can be found in Bringsjord and Heuveln (2003), and in Cook (forthcoming). For more about hypercomputers, see Bringsjord (1998), and Bringsjord and van Heuveln (2003).

[3] But this doesn't mean that any infinite steps, each one being a premise or obtain from the previous ones by one admissible rule, is automatically a "proof". Call this sense "$proof_3$". Suppose that there is a language that talks about sets, and that has a name to for set. Suppose further that for each name 'a', I have a prove of A(a). So it's valid to infer "“∀A(x)" from all those premises, but no machine –I mean, no hypercomputer-can carry out that "$proof_3$". So thinking of a "proof" as whatever a machine can carry out doesn't put as in an ideal position –because there are still valid inferences without a proof. But that doesn't mean that we are not in a better position. We do have proofs of inferences that are not provable in the classic conception of what a "proof", we do have proofs with infinite steps. In a nutshell: it's not perfect, but it's better.

and a machine, in both cases, is something that can be defined by a finite set of instructions, but (2) it is a more general approach, that extends the finitary sense of "proof": all classical proofs are proofs in the new sense, but some things that are not proofs in the classical sense are proofs in the new sense of "proof." But it also provides an answer to the question of why we used to relate "proof" to an operation carried out in a finite number of steps. A proof is a mechanical operation. We used to think that mechanical operations are inevitably finite. But now we realize that they are not. So now we face two options: (a) to retain the link between "proof" and a "mechanical operation," or (b) to restrict it to a "mechanical finite operation." But why should we do that? Maybe because we want to associate the idea of "proof" with "what in principle a human being can do." However –at least if one thinks that what a human can calculate is no more than what a Turing Machine can do– as Bringsjord and van Heuveln (2003) say, that is not an obvious thesis. (And in fact, Bringsjord thinks that it is a false one.) And it makes the idea of "proof," a central idea in Logic, and epistemic notion. That seems an unpleasant result, at least if one thinks that Logic must be independent of what human beings can or cannot do.

But the "finitist" position –the one that defends a finitist notion of "proof"– has further undesirable consequences, such as the following: in second order logic some inferences with an infinite number of premises are valid, in a semantic sense, because there is no valuation that makes all premises true, and the consequence, false. But there is no finite proof of it, because the hypothetical proof would have an infinite number of steps –at least, because it does have an infinite number of premises. (But in any case, the semantic apparatus is more powerful than any finitist syntaxes.)

Another argument in favor of infinitary systems is the one presented by Cook himself: if our theory about truth –and, in particular, that part of it that explains paradoxes– does not *also* apply to infinitary systems, which may be spoken and understood by rational beings very much like ourselves, but with the ability to "see" infinite lists of sentences (or infinitely long sentences), then the theory of truth we are talking about would be reduced to a theory of truth in English, Spanish, or some other subcategory of truth *simpliciter*. Priest (1997) and Beall (2001) have defended the idea that we, as finite reasoners –that is, as entities that have at most the computational skills of a Turing Machine–, cannot understand directly what an infinite sentence claims, nor can we directly process any infinite amount of information. Bringsjord and van Heuveln claim that neither Priest nor Beall have presented conclusive reasons to accept our alleged limitations. But Cook notes that, even if Priest and Beall are right, that will not be a reason against infinitary systems; unless those limitations are universal, that is, (probably necessarily) shared by all rational beings. But Cook seems to have been a little too lenient with his rivals. Let's suppose that, indeed, those limitations are universal. (I mean: those are the limitations of any rational being, and not only our particular situation.) Why should we base our theory of truth on those epistemic limitations? Our theory of truth is not supposed to be epistemic in any relevant sense, not even in this one. There will still be inferences in those systems that preserve truth. And a theory of truth must explain why.

One last way to defend infinitary systems is by comparing them with second-order finitary systems with standard semantics. Xxx Infinitary-quantifier languages such as $L(\omega_1, \omega_1)$ share with second-order languages defects like incompleteness, and virtues like their strong expressive power. One could reject second-order logic as a "real" logic. But an analogous maneuver is not so obvious when we are talking about systems designed to express arithmetic truths. The resemblance between second-order systems of this type and some infinitary systems as the ones we are referring to seems to give extra legitimacy to the latter.[4]

2. The status of Yablo's list

Probably the best thing that Priest and Beall can claim in favor of the idea that Yablo's paradox is really a circular paradox is that they –but neither Yablo nor Cook– can prove that the list exists. Not only do they *postulate* the list –as Yablo and Cook do–, but they also prove the list's existence.

There are two ways to read the Priest's assertions quoted by Barrio in his contribution to this volume (in §2). In the first place, what Priest claims is that we do not have any guarantee that Yablo's predicate exists. Indeed Teijeiro, in her contribution to this volume, defends this same position. But whether or not this is a relevant reading of Priest's version of Yablo's list, it is surely not a good one of Cook's version of it. Each S_m sentence of Yablo's list, in Cook's infinitary system, is just as follows:

[4] In fact, in a sense, Lp is weaker than $L(\omega_1, \omega_1)$, because it doesn't allows formulas with infinite quantifiers. (In fact, Lp doesn't have quantifiers in it's vocabulary at all.) And, in a way, it is a sub-classical system. ("In a way", because it has predicates 'F' and 'T'.) So, if a stronger infinity language is admitted, why won't Lp's language be so?

$\delta(S_m) = \wedge\{F(S_n) : n > m \;\&\; n, m \in \omega\} = F(S_{m+1}) \wedge F(S_{m+2}) \wedge F(S_{m+3}) \wedge F(S_{m+4})\ldots$

So, as one can see, this is just an infinite conjunction, and each conjunct is the False Predicate applied to a name of a sentence. These elements are just part of the vocabulary, and any of these sentences is just a well-formed formula of the language. So, any acceptable assignment –i.e., any valuation– must give each sentence in the list a stable truth value. But there are no such assignments. Besides, it is possible to derive a contradiction in δD in this list. So Yablo's list is both semantically and syntactically paradoxical. There is nothing problematic with the analogue of Priest's version of Yablo's predicate –i.e., the predication of falsity to each of "the following" sentences. And we have not "postulated" the list, but derived a contradiction by $\omega^2 + 3$ applications of the rules of the δD system.

The second way to interpret Priest's claim (a position that Beall develops) is that he –but neither Yablo nor Cook– can prove that the list exists, because each line in the list is a theorem of PA with enough resources to diagonalize. (Each line will be the result of applying I∀ to UFYP). What Yablo and Cook do is *postulate* the existence of the list.[5] But, as Priest's quote asserts, "we can imagine all sorts of things that do not exist".

Nevertheless, neither Yablo nor Cook need that each sentence of their respective "Yablo's list" (formulated in a finitary or in an infinitary system) be a theorem of the relevant system, in order to prove that the list really exists. Each sentence in the list is a well-formed formula of the language, and each valuation or assignment should give it a truth value. The problem is that it is not possible to do that, because the assignment will not be stable. True: we do need a denotation function δ such that for each S_n, $\delta(S_n) = \wedge\{F(S_n) : n > m \;\&\; n, m \in \omega\}$. In order to get a paradox from it, we need to fix a pattern of reference. That's function δ's job. With it, we have a "paradox," in the semantic sense identified above. In fact, we have a proof that Cook's version of Yablo's paradox is syntactically paradoxical. But to be paradoxical, it is just enough to be paradoxical in the semantic sense.

Does the use of function δ make it fair to say that Cook has postulated the list? It depends. It's true that the pattern of reference that

[5] The problem is not whether that predicate exists. It does: it only uses the truth –or the satisfaction– predicate, and some arithmetical functions. And it is a well-formed predicate.

it fixes is not necessary: there may be others. Actually: there are others. This particular δ is just a function. The whole S_n form a countable set, and so do the sentences they refer to. There seems nothing suspicious in this particular function.[6] But with it, we have a paradox.

This is the case of the "Old-Fashioned Liar": it is a sentence a first order language capable of expressing identity (and not just equivalence) over sentences,[7] and each valuation should assign a truth value to it. The "Old-Fashioned Liar" is a paradigmatic example of a circular sentence that generates a paradox. But it is not a theorem –or at least it is not of a theory like PA plus all instances of the T-schema. If one rejects the idea that Yablo's list generates a paradox, just because it is not the case that each sentence of the list is a theorem, then one should also reject the idea that the "Old-Fashioned Liar" is also a sentence that generates a paradox. But that seems like a very extreme position. In each case, the principles that allow the formation of sentences (i.e., the Liar and each sentence in Yablo's list) are plausible. But there is no stable assignment of truth values to them.

3. Conclusion

There are two senses in which a set of sentences can be a paradox: the semantic and the syntactic ones. Cook has presented proofs that it is possible to formulate an infinitary version of Yablo's list that is paradoxical in both senses, and that is also non-circular. The main reason for rejecting this idea is to argue against infinitary systems. But it is not possible to do so without giving up a general theory of truth. The other way to argue against the idea that Yablo's list is truly paradoxical is to

[6] Remember what Kripke says in defense of self-referentiality: "A simpler, and more direct, form of self-reference uses demonstratives or proper names: Let 'Jack' be a name of the sentence 'Jack is short'… I can see nothing wrong with "direct" self-reference of this type. If 'Jack' is not already a name in the language, why can we not introduce it as a name of any entity we please? In particular, why can it not be a name of the (uninterpreted) finite sequence of marks 'Jack is short'? (Would it be permissible to call this sequence of marks "Harry," but not "Jack"? Surely prohibitions on naming are arbitrary here.) There is no vicious circle in our procedure, since we need not *interpret* the sequence of marks 'Jack is short' before we name it. Yet if we name it "Jack," it at once becomes meaningful and true" (Kripke 1975, p. 693). Function δ name some sentences of the language. All those expressions and are already part of the vocabulary. So, what might be wrong with a function like it?

[7] That may include natural languages, if they admit valuations, if the sentence "This sentence is false", that says about itself that it is false, is of that sort. But, as I see it, the 'Old Fashioned Liar' may be no less formal than the 'Arithmetical Liar'.

claim that none of the sentences in the list is a theorem. But if that is right, then the classical example of paradox, the "Old-Fashioned Liar," will not be a paradox either. And that seems to be an undesirable consequence.

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